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INTRODUCTION

An important part of the recently developed potential theory required especially by the connections between the probability theory and the potential theory, is the study of the cone of excessive functions with respect to a resolvent family of kernels on a measurable space. This study was initiated by G.A. Hunt in 1957 which introduced and treated also the concept of excessive measures.

In the present work we intend to unify into a single theory these two directions, namely the study of excessive functions and that of excessive measures which were treated up to now separately. For this purpose, we introduce the concept of \( \mathcal{E} \)-lattice cone, which has sufficiently many properties in order to give the possibility to develop a theory of resolvent families of kernels and of the corresponding potential theory. Also, this concept is sufficiently large in order to contain both functions and measures as well as other important examples in applications as for instance stochastic processes.

The book is divided into 6 chapters. In chapter 1 are introduced and studied the fundamental concepts of \( \mathcal{E} \)-lattice cone and kernel. The réduite, the fundamental algebraic and lattice properties of the cone of excessive elements are presented in chapters 2 and 3. The infinitesimal operator associated with a resolvent, the domination principle and Hunt's theorem for the existence of a resolvent are extensively treated in this general framework in chapters 3 and 4. In chapter 5 are introduced and studied a dual theory and the concept of energy. Finally, in chapter 6, the whole theory is applied to the stochastic processes where it is shown that the right continuous supermartingales may be regarded as excessive elements with respect to a convenient resolvent family of kernels.

Up to the chapter 6 the text is selfcontained, except for some elementary notions concerning ordered sets and vector lattices. Also, for a better understanding of some examples and exercises and especially for chapter 6, elementary notions on measure theory and on stochastic processes are required.

Examples and further developments are presented in exercises.
CHAPTER 1. $\mathcal{C}$-LATTICE CONES

1. DEFINITIONS AND PRELIMINARY RESULTS

Throughout this book we shall denote by $\mathbb{N}$, the set of all strictly positive integers, by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}_+$ the set of all nonnegative real numbers, endowed with their natural algebraic and order structures. For the set of strictly positive numbers, we shall use the notation $(0, \infty)$.

If $X$ is a set, we shall denote simply by $(x_n)$ a sequence in $X$.

If $(X, \leq)$ is an ordered set, then the least upper (respectively, the greatest lower) bound of a set $A$ or of family $\{x_i | i \in I\}$ of $X$ will be denoted as usual by $\vee A$ or $\bigvee_{i \in I} x_i = \vee x_i = \vee x_i$ (respectively, $\wedge A$, ...) if it does exist. If $J = \{1, 2\}$ we write $x_1 \vee x_2$ (respectively, $x_1 \wedge x_2$). If $X = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, we use sup (respectively inf) instead of $\vee$ (respectively $\wedge$). The ordered set $X$ is called a lattice (respectively, a $\mathcal{C}$-complete lattice, respectively a complete lattice) if $\vee A$ and $\wedge A$ exist for any finite (respectively countable, respectively arbitrary) nonempty set $A$. It is called conditionally lower $\mathcal{C}$-complete (respectively, conditionally lower complete) if $\wedge A$ exists, for any countable (respectively, arbitrary) nonempty lower bounded set $A$. Analogously, for conditionally upper $\mathcal{C}$-complete (respectively, conditionally upper complete). We say that $X$ is conditionally $\mathcal{C}$-complete (respectively, conditionally complete) when it is both upper and lower conditionally $\mathcal{C}$-complete (respectively, upper and lower conditionally complete).

If $(X, \leq)$ is an ordered set and $A \subseteq X$, we say that $A$ is a solid, if for any $x, y, z \in X$, such that $x \leq y \leq z$ and $x, z \in A$, we have $y \in A$.

A set $C$ endowed with an addition and a multiplication with non-negative real numbers will be called a convex cone if the following axioms are satisfied:

1) $(x+y)+z = x+(y+z)$
2) $x+y = y + x$
3) there exists an element denoted by $0_C = 0$ such that $x + 0 = x$
4) $\alpha(x+y) = \alpha x + \alpha y$
5) $(\alpha + \beta)x = \alpha x + \beta x$
6) $\alpha(\beta x) = (\alpha \beta)x$
7) \( l \cdot x = x \)
8) \( 0 \cdot x = 0 \cdot 0 = 0 \)

In the above relations \( x, y, z \) are elements of \( C \) and \( \alpha, \beta \) are nonnegative real numbers.

A convex cone \( C \) endowed with an order relation "\( \leq \)" is called an ordered convex cone, if for any \( x, y \in C \) such that \( x \leq y \), we have \( x + y \leq y + z \) for any \( z \in C \) and \( \alpha x \leq \alpha y \) for any nonnegative real number \( \alpha \).

We see that \( \mathbb{R}, \mathbb{R}^+ \) and \( \overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\} \), endowed with their natural algebraic structures and order relations are ordered convex cones.

An ordered convex cone is called a convex cone of positive elements if: a) for any \( x \in C \), we have \( x \geq 0 \) b) for any \( x, y \in C \) such that \( x \leq y \) there exists \( z \in C \) such that \( x + z = y \).

Let \( C \) be a convex cone. For \( x, y \in C \) denote \( x \leq' y \) if there exists \( z \in C \) such that \( x + z = y \). It is immediately seen that "\( \leq' \)" is a preorder relation and if \( C \) is a convex cone of positive elements, we have \( x \leq y \) if and only if \( x \leq' y \). Hence, a convex cone of positive elements is nothing else but a convex cone endowed with the relation "\( \leq' \)", provided that this is an order relation. In order that the relation "\( \leq' \)" on a convex cone be an order relation, it is sufficient that the following properties hold: \( x + z = y + z \Rightarrow x = y \) and \( x + y = 0 \Rightarrow x = y = 0 \).

If \( C \) is a convex cone of positive elements and \( A \) is a subset of \( C \), containing the null element of \( C \), then \( A \) is solid if and only if for any \( x, y \in C \), such that \( x \leq y \) and \( y \in A \), we have \( x \in A \).

A convex cone of positive elements \( C \), is called a \( \mathcal{G} \)-lattice cone if the ordered set \( C \) is a \( \mathcal{G} \)-complete lattice and if for any \( x \in C \) and any sequence \( (x_n) \) in \( C \), the following distributivity relations hold:

\[
\begin{align*}
    x \lor (\land x_n) &= \land (x \lor x_n) \\
    x \land (\lor x_n) &= \lor (x \land x_n) \\
    x + \land x_n &= \land (x + x_n) \\
    x + \lor x_n &= \lor (x + x_n)
\end{align*}
\]

We give now some examples of \( \mathcal{G} \)-lattice cones.

1) If \( E \) is an arbitrary set, \( \mathbb{R}_E^+ \), endowed with the pointwise algebraic operations and order relation.

2) If \( (E, \mathcal{E}) \) is a measurable space, \( \{f \mid f \in \mathbb{R}_E^+, f \ \mathcal{E}-\text{measurable}\} \) endowed with the pointwise algebraic operations and order relation.
3) If \((E, \mathcal{F}, \mu)\) is a measurable space, \(\{\mu \mid \mathbb{R}^+_+, \mu \text{ measure}\}\) endowed with pointwise algebraic operations and order relation (a proof that this is a \(\sigma\)-lattice cone, can be found in the Theorem 1.2.1).

4) If \((E, \mathcal{F}, \mu)\) is a \(\sigma\)-finite measure space, the set of equivalence classes modulo \(\mu\) of positive numerical measurable functions.

5) \(\prod_{i \in I} C_i\), where \(I\) is an arbitrary index set and \(C_i\), \(i \in I\) are \(\sigma\)-lattice cones.

6) If \((\Omega, \mathcal{T})\) is a measurable space and \((\mathcal{T}_t)_{t \in T}\) \((T\) is an arbitrary ordered set) is an increasing family of sub-\(\sigma\)-algebras of \(\mathcal{T}\), the set of stochastic positive processes adapted to this basis.

Throughout this book, \(C\) will be a \(\sigma\)-lattice cone.

If \((x_n)\) is a sequence in \(C\), we shall denote

\[\sum_{n=1}^{\infty} x_n = \bigvee_{n=1}^\infty x_n = \bigwedge_{k=1}^n x_k\]

\[\liminf_{n \to \infty} x_n = \bigvee_{n \to \infty} x_n = \bigwedge_{k \geq n} x_k\]

and \(\limsup_{n \to \infty} x_n = \bigwedge_{n \to \infty} x_n = \bigvee_{k \geq n} x_k\).

When \(\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n = x\), we say that \((x_n)\) is convergent (to \(x\)) and we denote \(x = \lim x_n\), or \(x_n \to x\). Moreover, if \((x_n)\) is increasing (respectively, decreasing) we write \(x_n \nearrow x\) (respectively \(x_n \searrow x\)).

If \((x_n)\) and \((y_n)\) are two sequences in \(C\) and \(\alpha \in \mathbb{R}_+\), the following relations are immediate from the definitions:

\[\alpha (\bigvee x_n) = \bigvee \alpha x_n\], \[\alpha (\bigwedge x_n) = \bigwedge \alpha x_n\]

\[\liminf x_n + \liminf y_n \leq \liminf (x_n + y_n) \leq \liminf x_n + \limsup y_n\]

\[\limsup x_n \leq \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n\]

These inequalities are true also when we replace + by \(\vee\) or \(\wedge\). Particularly, if \(x_n \to x\) and \(y_n \to y\), we have \(x_n + y_n \to x + y\), \(x_n \vee y_n \to x \vee y\) and \(x_n \wedge y_n \to x \wedge y\). Also, if only \((x_n + y_n)\) is convergent, then

\[\lim (x_n + y_n) = \liminf x_n + \limsup y_n\]
Proposition 1.1.1. Let \( x \leq C \) and let \((x_n)\) and \((y_n)\) be two sequences in \( C \).

a) If \( x \leq \sum x_n \), then there exists a sequence \((u_n)\) in \( C \) such that \( x = \sum u_n \) and \( u_n \leq x_n \), for any \( n \in \mathbb{N} \).

b) \( x \wedge (\sum x_n) \leq \sum x \wedge x_n \)

c) If \( x = x_n + y_n \), for any \( n \in \mathbb{N} \), then \( x = \bigvee x_n + \bigwedge y_n \)

Proof. Let us denote \( u_1 = x \wedge x_1 \). Further, for any \( V_n \in C \), such that
\[
x \wedge (x_1 + \ldots + x_n + x_{n+1}) = x \wedge (x_1 + \ldots + x_n + x_{n+1})
\]
and denote \( u_{n+1} = x_{n+1} \wedge v_n \). Now, we may prove by induction that for any \( n \in \mathbb{N} \), we have
\[
x \wedge (x_1 + \ldots + x_n) = u_1 + \ldots + u_n
\]
Indeed, this equality is obvious for \( n=1 \). Assuming it true for \( n \), we get
\[
x \wedge (x_1 + \ldots + x_n + x_{n+1}) = [x \wedge (x_1 + \ldots + x_n) + v_n] \wedge [x \wedge (x_1 + \ldots + x_n) + x_{n+1}]
\]
= \( x \wedge (x_1 + \ldots + x_n) + v_n \wedge x_{n+1} = u_1 + u_2 + \ldots + u_n + u_{n+1} \)

Taking the supremum in the above equality, the proof is complete.

b) follows immediately from a)

c) We prove first that
\[
x_1 \vee x_2 + y_1 \wedge y_2 = x
\]
We have
\[
x_1 \vee x_2 + y_1 \wedge y_2 = (x_1 + y_1 \wedge y_2) \vee (x_2 + y_1 \wedge y_2) =
\]
\[
= [(x_1 + y_1) \wedge (x_1 + y_2)] \vee [(x_2 + y_1) \wedge (x_2 + y_2)] \leq x \vee x = x
\]
On the other hand
\[
x_1 \vee x_2 + y_1 \wedge y_2 = (x_1 \vee x_2 + y_1) \wedge (x_1 \vee x_2 + y_2) =
\]
\[
= [(x_1 + y_1) \vee (x_2 + y_1)] \wedge [(x_1 + y_2) \vee (x_2 + y_2)] \leq x \wedge x = x
\]
By induction, we have
The general relation follows now by convergent arguments.\footnote{The proof of this statement requires a detailed analysis of the properties of the vector lattice.}

Remark. For any two elements $x, y \in C$, we have

$$x + y = x \vee y + x \wedge y$$

which is an elementary relation in vector lattices.

For any element $x \in C$ we shall denote by

$$x = \bigwedge_{n \in \mathbb{N}} \frac{1}{n} x$$

Obviously we have : $x = \bigwedge \{ \alpha x \mid \alpha \in (0, \infty) \}$, $\alpha x = x$ for any real number $\alpha > 0$, $\overline{x} = x$, $x + y = x + y$. Further, if $x \leq y$, then $x + y = y$.

Indeed, there exists $z$ such that $x + z = y$, hence $y + x = 2x + z = x + z = y$.

Proposition 1.1.2. If $(\alpha_n)$ is a sequence in $\mathbb{R}_+$ converging to $\alpha > 0$ and $(x_n)$ is a sequence in $C$ converging to $x$, then the sequence $(\alpha_n x_n)$ is convergent to $\alpha x$.

Proof. Obviously, for any $\beta \in \mathbb{R}_+$, we have $\beta x_n \rightarrow \beta x$, hence for any $\varepsilon > 0$ and $\varepsilon < \alpha$, we have

$$(\alpha - \varepsilon)x \leq \lim \inf \alpha_n x_n \leq \lim \sup \alpha_n x_n \leq (\alpha + \varepsilon)x$$

It remains only to show that

$$\bigvee_{\varepsilon > 0} (\alpha - \varepsilon)x = \alpha x = \bigwedge_{\varepsilon > 0} (\alpha + \varepsilon)x$$

Indeed, from $x = \varepsilon x + (\alpha - \varepsilon)x$ and $(\alpha + \varepsilon)x = \alpha x + \varepsilon x$, using Proposition 1.1.1(c), we get

$$x = \overline{x} + \bigvee_{\varepsilon > 0} (\alpha - \varepsilon)x, \quad \bigwedge_{\varepsilon > 0} (\alpha + \varepsilon)x = \alpha x + \overline{x}$$

Now, since $\overline{x} \leq \beta x$ for any $\beta > 0$, we get using the properties of $\overline{x}$,

$$\overline{x} + \bigvee_{\varepsilon > 0} (\alpha - \varepsilon)x = \bigvee_{\varepsilon > 0} (\alpha - \varepsilon)x$$

and $\alpha x + \overline{x} = \alpha \overline{x}$.

Let $x$ be an element of $C$. Then we know that for any $y \in C$, $y \geq x$ there exists $z \in C$, such that $x + z = y$. The element $z$, as one can easily see from examples, is not unique, generally. Those elements $x$, for which the uniqueness holds, will play an important role in further developments of the theory. Therefore we introduce the following definition:

An element $x \in C$ is called **subtractable** if for any $y \in C$, $y \geq x$, there exists uniquely $z \in C$ such that $x + z = y$. In this
case the notation \( z = y - x \) makes sense and in fact it suggested the term subtractible.

In the case of functions or of classes of functions, subtractible means obviously finite, respectively finite almost everywhere. For measures, it is easy to see that subtractible means \( \mathcal{G} \)-finite (see also Theorem 1.2.1).

**Proposition 1.1.3.** An element \( x \in \mathcal{C} \) is subtractible if and only if one of the following conditions holds:

a) for any \( y, z \in \mathcal{C} \) such that \( x + y \leq x + z \), we have \( y \leq z \)

b) \( x = 0 \)

**Proof.** Assume that \( x \in \mathcal{C} \) is subtractible. From \( x + x = x \) we get \( x = 0 \).

Assume now b) satisfied and let \( y, z \in \mathcal{C} \) be such that \( x + y \leq x + z \). By induction, we may prove that for any \( n \in \mathbb{N} \) we have \( x + ny \leq x + nz \), hence \( y \leq z \).

Obviously, if a) holds \( x \) is subtractible.

We shall denote by \( \mathcal{C}_s \) the set of all subtractible elements of \( \mathcal{C} \).

Using the above proposition we see that \( \mathcal{C}_s \) is a solid subcone of \( \mathcal{C} \).

An important case, which is often found in the most interesting examples of this theory, is that when every element of \( \mathcal{C} \) is the supremum of a sequence of elements from \( \mathcal{C}_s \). In this case we say that \( \mathcal{C} \) is a proper \( \mathcal{G} \)-lattice cone.

We shall denote by \( |\mathcal{C}| \) the equivalence classes of \( \mathcal{C} \times \mathcal{C}_s \) with respect to the equivalence relation

\[
(x, y) \sim (x', y') \quad \text{if} \quad x + y' = y + x'
\]

Further, for any \( (x, y) \), we denote by \( |x, y| \) its equivalence class. The set \( |\mathcal{C}| \) becomes a convex cone with respect to the following operations

\[
|x, y| + |u, v| = |x + u, y + v| \quad \text{and} \quad \alpha|x, y| = |\alpha x, \alpha y|
\]

Using the definition of \( \mathcal{C}_s \), we may see that the relation

\[
|x, y| \leq |u, v| \quad \text{if} \quad x + v \leq y + u
\]

is an order relation on \( |\mathcal{C}| \). Thus, \( |\mathcal{C}| \) is an ordered convex cone. We shall identify any element \( x \) of \( \mathcal{C} \) with the class \( |x, 0| \) from \( |\mathcal{C}| \) and thus \( |\mathcal{C}| \) becomes an ordered convex subcone of \( |\mathcal{C}| \). If the element \( |x, y| \) from \( |\mathcal{C}| \) is such that \( x \in \mathcal{C}_s \), we have \( |x, y| + |y, x| = 0 \).
In this situation, we shall denote $|y, x|$ by $-|x, y|$. Thus, for any $|x, y| \in C$, we have

$$|x, y| = |x, o| + |o, y| = |x, o| - |y, o|$$

According to the above identification convention, we shall write $|x, y| = x - y$, i.e.

$$|C| = C - C_s$$

In the sequel we shall drop out completely the class notation $|x, y|$ and use instead $x - y$.

We shall show that $|C|$ possesses several lattice properties. We remark first that if $u, v$ are two elements in $|C|$, we have $u \leq v$ if and only if there exists $z \in C$ such that $u + z = v$. Let now $(u_i)_{i \in I}$ be a nonempty countable family in $|C|$ and assume that there exists $z \in C_s$ such that $u_i + z = x_i \in C$ for any $i \in I$. Then the elements $\bigvee x_i - z$ and $\bigwedge x_i - z$ are exactly the supremum and respectively the infimum of the family $(u_i)_{i \in I}$. Since the above condition holds for any finite, respectively increasing, respectively decreasing and minorated family in $|C|$, we deduce first, that $|C|$ is a lattice, which is upper-\(\mathcal{G}\)-complete and conditionally lower-\(\mathcal{G}\)-complete. Analogously, one may prove the distributivity in $|C|$ of the operation $+$, $\bigvee$, $\bigwedge$ with respect to countable $\bigvee$ and $\bigwedge$ (see the relations in the definition of a \(\mathcal{G}\)-lattice cone, page 2).

For any element $x$ from $|C|$, we denote $x^+ = x \lor o$ and $x^- = x \land o$. We have $x^- \in C_s$, $x^+ \land x^- = o$ and $x = x^+ - x^-$. If $x = y - z$, $y \in C$ and $z \in C_s$, then we have $y \geq x^+$ and $z \geq x^-$. Moreover, if $y \land z = o$, then we have $y = x^+$ and $z = x^-$, hence the above decomposition is unique.

Though $|C|$ is not a vector lattice, some familiar relations concerning $x^+$ and $x^-$ are still valid: $(x+y)^+ \leq x^+ + y^+$, $(x+y)^- \leq x^- + y^-$. We may also introduce the notation $|x| = x^+ + x^-$ and we have $|x+y| \leq |x| + |y|$. The set $C_s - C_s$ in $|C|$ is a conditionally \(\mathcal{G}\)-complete vector lattice whose positive elements coincide with $C_s$.

Let $C$ and $D$ be two \(\mathcal{G}\)-lattice cones. A map $V : C \rightarrow D$ is called a kernel on $(C,D)$, if

1) $V o = o$

2) for any sequence $(x_n)$ from $C$
\[ V(\sum_{n} x_n) = \sum_{n} Vx_n \]

If \( D = C \), we say simply, kernel on \( C \).

Remark. In the above definition, the conditions 1) and 2) are equivalent with the following ones:

1') \( V0 = 0 \)

2') \( V(x+y) = Vx + Vy \) for any \( x, y \in C \)

3') for any increasing sequence \( (x_n) \) from \( C \), we have

\[ V(\bigvee_{n} x_n) = \bigvee_{n} Vx_n \]

Using Proposition 1.1.2 we may prove that for any \( \alpha \in \mathbb{R}_+ \) and \( x \in C \), we have

\[ V(\alpha x) = \alpha Vx \]

Also, if \( (x_n) \) is a decreasing sequence in \( C \) such that \( Vx \) is subtractible (in \( D \)), we have

\[ V(\bigwedge_{n} x_n) = \bigwedge_{n} Vx_n \]

Indeed, using Proposition 1.1.1c), we may construct an increasing sequence in \( C \), \( (y_n) \) such that \( x = y_n + x_n \) for any \( n \in \mathbb{N} \). Hence, we have

\[ V(\bigvee_{n} y_n) + V(\bigwedge_{n} x_n) = Vx = Vy_n + Vx_n = \bigvee_{n} Vy_n + \bigwedge_{n} Vx_n = V(\bigvee_{n} y_n) + \bigwedge_{n} Vx_n \]

If \( W \) is a kernel on \( (D,E) \), where \( E \) is another \( \mathcal{G} \)-lattice cone, then the map \( x \rightarrow W(Vx) \), \( x \in C \) is a kernel on \( (C,E) \) which will be denoted by \( WV \).

If \( (V_n) \) is a sequence of Kernels on \( (C,D) \), we see that the map \( V: C \rightarrow D \) defined by \( Vx = \sum_{n} V_n x \) is again a kernel and will be denoted by \( \sum V_n \). For any \( \alpha \in \mathbb{R}_+ \) and any kernel \( V \), we denote by \( \alpha V \) the kernel \( (\alpha V)x = \alpha Vx \). If \( V_1, V_2 \) are two kernels, we write \( V_1 \leq V_2 \), if for any \( x \in C \), \( V_1 x \leq V_2 x \). With these definitions, the set of kernels on \( (C,D) \) becomes an ordered convex cone (the null element being the map which assigns to every \( x \in C \), the null element of \( D \)), which may have some lattice properties. The question whether it is a
\( C \)-lattice cone is not solved generally. The answer to this question in the particular case when \( D = R^+ \) is solved in Theorem 1.2.1.

A kernel \( V \) on \((C,D)\) is called proper if for any \( x \in C \), there exists a sequence \((x_n)\) in \( C \), increasing to \( x \) such that \( Vx_n \in D_s \) for any \( n \in \mathbb{N} \).

According to this definition, a \( C \)-lattice cone \( C \) is proper if and only if the identity map is a proper kernel on \( C \). It is easy to see that the sum of two proper kernels is also a proper kernel and any kernel dominated by a proper kernel is also proper.

**Proposition 1.1.4.** Let \((V_n)\) be a sequence of kernels on \((C,D)\).

a) If \((V_n)\) is increasing, then there exists \( \bigvee V_n \) and for any \( x \in C \), we have
\[
(\bigvee V_n) x = \bigvee V_n x
\]

b) If \((V_n)\) is decreasing and \( V_1 \) is proper, then there exists \( \bigwedge V_n \) and for any \( x \) such that \( V_1 x \in D_s \), we have
\[
(\bigwedge V_n) x = \bigwedge V_n x
\]

**Proof.**

a) The map \( x \mapsto \bigvee V_n x \) is a kernel which is in fact \( \bigvee V_n \).

b) Denote \( C_0 = \{ x \in C \mid V_1 x \in D_s \} \) and \( Vx = \bigwedge V_n x \) for any \( x \in C_0 \). The map \( V : C_0 \to D_s \) is obviously additive and for any decreasing sequence \((x_n)\) in \( C_0 \), we have \( V(\bigwedge x_n) = \bigwedge Vx_n \), hence, if \((y_n)\) is increasing to an element \( x \) belonging to \( C_0 \), we have \( Vx = \bigveeVy_n \).

Indeed, one may chose a decreasing sequence \((z_n)\) in \( C \) such that \( x = y_n + z_n \), for any \( n \in \mathbb{N} \). Obviously, \((z_n)\) is in \( C_0 \). Thus we get
\[
x = \bigvee y_n + \bigwedge z_n
\]
and
\[
Vx = V(\bigvee y_n) + V(\bigwedge z_n) = V(y_n) + V(z_n) = \bigvee V(y_n) + \bigwedge V(z_n) = \bigvee V(y_n) + V(\bigwedge z_n)
\]
\[
= \bigvee V(\bigwedge z_n).
\]

Since \( V(\bigwedge z_n) \in D_s \), we deduce the above equality.

Let now \( x \) be arbitrary in \( C \). Since \( V_1 \) is proper, there exists a sequence \((x_n)\) in \( C_0 \), increasing to \( x \). Denote \( Vx = \bigvee Vx_n \). From the above proof, one may easily see that this definition does not depend on the sequence \((x_n)\). It is easy to see that \( V \) is a kernel on \((C,D)\) which is in fact \( \bigwedge V_n \).
A kernel $K$ on $C$ is called semi-indicator, if it satisfies the following conditions:

a) $K \propto I$, where $I$ is the identity map on $C$

b) $KK = K$

c) for any sequence $(x_n)$ of elements of $C$

$$K(\vee x_n) = \vee K(x_n) \quad \text{and} \quad K(\wedge x_n) = \wedge K(x_n)$$

Obviously, if $C$ is proper, any semi-indicator is a proper kernel. The property a) from the above definition enables us to extend linearly any semi-indicator $K$ on $|C|$ taking $Kx = Kx^+ - Kx^-$ for any $x \in C$. One can easily see that this extension is an additive and positively homogeneous map from $|C|$ into $|C|$. Also, the property b) and c) from the above definition holds, provided that $\wedge x_n$ makes sense in $|C|$. In the sequel we shall assume that any semi-indicator is defined on $|C|$.

For any $x \in C$, we denote $\bar{x} = \sqrt{x}$ and $I_x$ the map $y \rightarrow \bar{x} \wedge y$, i.e.

$$I_x(y) = \vee_{n} (nx \wedge y) \quad \text{for any} \ y \in C$$

The map $I_x$ is additive: indeed, if $y, z \in C$, we have

$$\bar{x} \wedge (y+z) \leq \bar{x} \wedge y + \bar{x} \wedge z \leq 2\bar{x} \wedge (\bar{x}+y) \wedge (\bar{x}+z) \leq 2\bar{x} \wedge (y+z) = \bar{x} \wedge (y+z),$$

since $2\bar{x} = \bar{x}$. Also, the assertions a), b) and c) hold for $I_x$, hence $I_x$ is a semi-indicator. Obviously, we have $I_x = I_{\alpha x} = I_{\bar{x}}$, for any $\alpha > 0$ and if $x \leq y$, then $I_x \leq I_y$.

When $C$ is a cone of functions, we have

$$I_{\bar{x}}(y) = y 1_{\{x > 0\}}$$

where $1_{\{x > 0\}}$ denotes the characteristic function of the set on which $x$ is strictly positive.

In the abstract case the semi-indicators are intended to play the role of this operation, i.e. multiplication with a characteristic function. In this way, if $x_1, x_2 \in |C|$ we may use an intuitive terminology saying "$x_1 \preceq x_2$ on $\{x > 0\}$" if $I_{x_1} \leq I_{x_2}$.

**Proposition 1.1.5.** a) If $(x_n)$ is a sequence in $C$, then $\vee I_{x_n}$ does exist and

$$I_{\vee x_n} = \vee I_{x_n}$$
b) If $x \in C$ and $y \in C$, then

$$I_{(x-y)^+} x \geq I_{(x-y)^+} y$$

and

$$\bigvee_n I_{(nx-y)^+} = I_x$$

**Proof.** The first part from b) follows from the relations

$$x + (x-y)^- \geq y$$

and

$$I_{(x-y)^+} (x-y)^- = 0$$

a) Obviously, for any $n \in \mathbb{N}$, $I_{\bigvee_n x} \geq I_{\bigvee_n x}$ and if $V$ is a kernel such that $V \geq I_{\bigvee_n x}$, then $Vx \geq \bigvee_n x$, hence:

$$Vx \geq \bigvee_n (\bigvee_n x) = (\bigvee_n \bigvee_n x) \wedge x = (\bigvee_n \bigvee_n x) \wedge x = I_{\bigvee_n x} x$$

The second part of b) is an immediate consequence of a) and of the relation

$$\bigvee_n (nx-y)^+ = \bar{x}$$

Let us prove now this equality. The left hand side term is obviously smaller than the right hand side one. Let now $z$ be an element in $C$ such that $z \geq (nx-y)^+$ for any $n \in \mathbb{N}$. Then $z+y \geq nx \forall y \geq nx$ for any $n$, hence $z+y \geq \bar{x}$. Further, we have

$$\bar{x} = \frac{1}{n} x \leq \frac{1}{n} y + \frac{1}{n} z \leq \frac{1}{n} y + z$$

for any $n \in \mathbb{N}$. Making now $n$ tend to $\infty$, we get $\bar{x} \leq z$.

A kernel $K$ on $C$ will be called an indicator, if there exists a kernel $K'$, such that

a) $K+K' = I$

b) $KK' = K'K = 0$

Obviously, from this definition it results that $K^2 = K \leq I$ and that $K'$ is also an indicator. Further, we remark that $K'$ is uniquely defined. Indeed, if $K''$ is another kernel, satisfying the same conditions we have

$$K'' = K''K + K''K' = K''K' = K''K' + KK'' = K'$$
We shall call the kernel $K'$ the complement of the indicator $K$.

**Proposition 1.1.6.** Let $K$ be an indicator and let $x$ and $y$ be two elements of $C$. Then we have:

a) if $y \leq Kx$, then $Ky = y$

b) $K(x \wedge y) = Kx \wedge Ky = x \wedge Ky$

c) $I_{Kx} \leq K$ if $J$ is another indicator, then $KJx = JKx = Jx \wedge Kx$

d) $K(x \wedge y) \leq Kx \wedge Ky = K(Kx \wedge Ky) \leq x \wedge Ky = K(x \wedge y)$

Further, from

$$x \vee y = (Kx + K'y) \vee (Ky + K'y) \leq Kx \vee Ky + K'x + K'y$$

we obtain

$$K(x \vee y) \leq K(Kx \vee Ky) \leq Kx \vee Ky$$

The converse inequality is obvious.

c) $I_{Kxy} = \vee n (nKx \wedge y) = \vee n K(nx \wedge y) \leq Ky$

d) We have

$$KJx \leq Kx \wedge Jx = K(x \wedge Jx) \leq KJx$$

**Corollary 1.** Any indicator is a semi-indicator.

**Proof.** Indeed, if $K$ is an indicator and $(x_n)$ is a sequence in $C$, then using the assertion b) of the proposition and the fact that $K$ is a kernel, we have

$$K(\wedge x_n) \leq \wedge Kx_n \leq (\wedge x_n) \wedge Kx_1 = K(\wedge x_n)$$

by induction, we have

$$K(\bigvee_{i=1}^n x_i) = \bigvee_{i=1}^n K(x_i)$$

hence, since $K$ is a kernel

$$K(\bigvee_{i=1}^\infty x_i) = \bigvee_{i=1}^\infty K(x_i)$$

**Corollary 2.** For any two indicators $K$ and $J$, the kernel $KJ=JK$ is again an indicator.

**Proof.** It is easy to see that:

$$(KJ)' = K' + J'K$$
Proposition 1.1.7. Assume that $C$ is proper. Then any semi-indicator is an indicator.

Proof. Let $K$ be a semi-indicator. For any $x \in C_+$, denote

$$ K'x = x - Kx $$

It is easy to see that if $x = \sum x_n$, then $K'x = \sum K'x_n$. Since $C$ is proper, one may extend uniquely $K'$ to a kernel on $C$. Obviously, $K + K' = I$ and $K'K = 0$.

Proposition 1.1.8. Assume that $C$ is proper. Then, for any $x \in C$, there exists uniquely $x_f \in C_+$, such that

$$ x = x + x_f $$

and

$$ x \wedge x_f = 0 $$

Proof. We prove first the uniqueness. Assume $x + y = x + y'$ and $x \wedge y = x \wedge y' = 0$. Then, from $y \leq x + y'$, using Proposition 1.1.1 a), we get $y \leq y'$. Analogously we have $y' \leq y$.

Let us denote $I_x = K$. Since $C$ is proper, $K$ is an indicator and $x = Kx + K'x$. We have

$$ I_x x = \bigvee_n (nx \wedge x) = x \wedge x = x $$

and $K'x \in C_+$. Indeed,

$$ K'x = \bigwedge_n \frac{1}{n} K'x = K'(\bigwedge_n \frac{1}{n} x) = K'x = K'Kx = 0 $$

Also, using Proposition 1.1.6.d)

$$ x \wedge K'x = Kx \wedge K'x = KK'x = 0 $$

Remark. In the case of functions $x_f$ is the function equal to $x$ at any point at which $x$ is finite, and to 0 elsewhere; whereas $x_\infty$ is the function equal to $+\infty$, at any point at which $x$ is $+\infty$ and, to 0 elsewhere.

2. DUAL AND MULTIPLIERS

In this section we introduce and study a dual for a $\mathcal{G}$-lattice cone which will be also a $\mathcal{G}$-lattice cone. For the case of functions, the dual coincides with the cone of measures. Further, we shall study on a proper $\mathcal{G}$-lattice cone, the multipliers which are kernels commuting with any indicator and which in the case of functions, coincide with those kernels obtained by multiplication with a function.

Since, the dual will be essentially used only in the last two chapters (5,6) and the multipliers will be used only in sections...
2 and 3 of chapter 4, the reader may omit at first the lecture of this section.

The ordered convex cone of kernels on \((C, \mathbb{R}^+\)) will be called the **dual of \(C\)** and will be denoted by \(C^\star\). In the example 2) from section 1 the dual is example 3) and in the example 4), the dual is itself (via Radon-Nicodym theorem).

**Theorem 1.2.1.** The dual \(C^\star\) has the following properties:

a) \(C^\star\) is a \(\mathcal{S}\)-lattice cone and

\[
C^\star_s = \left\{ \mu \in C^\star \mid \mu \text{ proper kernel} \right\}
\]

b) \(C^\star\) is a complete lattice

c) For any \(\mu, \nu \in C^\star\) and any \(x \in C\), we have

\[
(\mu \vee \nu)(x) = \sup \{ \mu(y) + \nu(z) \mid y + z = x \}
\]

\[
(\mu \wedge \nu)(x) = \inf \{ \mu(y) + \nu(z) \mid y + z = x \}
\]

d) For any \(\mu, \nu \in C\) and any \(x \in C\), there exists \(y, z \in C\) such that \(y + z = x\) and

\[
(\mu \vee \nu)(y) = \nu(y) \quad (\mu \wedge \nu)(y) = \mu(y)
\]

\[
(\mu \vee \nu)(z) = \omega(z) \quad (\mu \wedge \nu)(z) = \nu(z)
\]

e) If \(\{ \mu_i \}\) is a family in \(C^\star\) and \(\mu \in C^\star\), then

\[
\mu + \nu \mu_i = \nu(\mu + \mu_i)
\]

\[
\mu + \nu \mu_i = \nu(\mu + \mu_i)
\]

\[
\nu \mu \wedge (\mu \nu \mu_i) = \mu \nu \mu_i
\]

**Proof.** Let us introduce first the following notation. If \(\mu \in C^\star\), denote

\[
C_\mu = \left\{ x \in C \mid \mu(x) < \infty \right\}
\]

and

\[
\overline{C}_\mu = \left\{ x \in C \mid \text{there exists a sequence } (x_n) \text{ in } C, \ x_n \to x \right\}
\]

As cone of kernels, \(C^\star\) is an ordered convex cone.

We prove now that \(C^\star\) is a positive convex cone. If \(\mu, \nu \in C\) are such that \(\mu \leq \nu\), we denote by \(\sigma\) the map defined on \(C_{\mu}\) by \(\sigma(x) = \nu(x) - \mu(x)\), where \(x \in C_{\mu}\). Obviously, \(\sigma\) is additive and if \((x_n)\) is a sequence in \(C_{\mu}\), \(x_n \to x\) and \(x \in C_{\mu}\), then we have \(\sigma(x_n) \to \sigma(x)\).

Further, we extend in a natural way \(\sigma\) on \(C\), putting \(\sigma(x) = \sup \{ \sigma(y) \mid y \in C_{\mu}, y \leq x \}\) if \(x \in C_{\mu}\) and \(\sigma(x) = +\infty\) if \(x \notin C_{\mu}\). We let it to the reader to verify that \(\sigma \in C^\star\) and \(\mu + \sigma = \nu\).
We start now to prove the lattice properties of $C^\kappa$. If $\mu, \gamma \in C^\kappa$, we denote for any $x \in C$

$$\lambda(x) = \sup \{ \mu(y) + \gamma(z) | y + z = x \}$$

and

$$\lambda'(x) = \inf \{ \mu(y) + \gamma(z) | y + z = x \}$$

Obviously $\lambda \geq \mu, \lambda \geq \gamma, \lambda' \leq \mu, \lambda' \leq \gamma$. Also, if $\mu \in C^\kappa$, is such that $\mu \geq \lambda$, $\mu \geq \gamma$ (respectively $\mu \leq \lambda, \mu \leq \gamma$), we have $\mu \geq \lambda$ (respectively, $\mu \leq \lambda$). If $x \leq y$, it is easy to see that $\lambda(x) \leq \lambda(y)$. Also we have $\lambda(x) \leq \lambda(y)$: indeed, let $y = y_1 + y_2$. Then, by Proposition 1.1.1a), there exists $x_1, x_2 \in C$ such that $x_1 + x_2 = x$, $x_1 \leq y_1$, $x_2 \leq y_2$ and therefore

$$\lambda(x) \leq \lambda'(x_1) + \lambda'(x_2) \leq \lambda'(y_1) + \lambda'(y_2)$$

As $y_1$ and $y_2$ are arbitrary, we get $\lambda(x) \leq \lambda'(y)$.

Let now $x, y$ be arbitrary in $C$. From the definitions, it is easy to see that

$$\lambda(x + y) \geq \lambda(x) + \lambda(y) \quad \text{and} \quad \lambda(x + y) \leq \lambda'(x) + \lambda'(y)$$

We prove now that $\lambda$ and $\lambda'$ are additive on $C^\kappa$. Let $x, y \in C\mu$ and $z_1, z_2 \in C$ be such that $x + y = z_1 + z_2$. Denote $x_2 = x \wedge z_2$ and take $x_1, y_2 \in C$ such that $x_1 + x_2 = x$, $x_2 + y_2 = z_2$, $x_1 \leq z_1$ and $y_2 \leq y$. Finally, chose $y_1$ and $z$ such that $y = y_1 + y_2$ and $z_1 = x_1 + z$. We have

$$x_1 + x_2 + y_1 + y_2 = x_1 + z + x_2 + y_2$$

$$\mu(x_1 + x_2 + y_2) + \mu(y_1) = \mu(x_1 + x_2 + y_2) + \mu(z)$$

hence $\mu(y_1) = \mu(z)$. Now

$$\mu(z_1) + \gamma(z_2) = \mu(x_1 + z) + \gamma(x_2 + y_2) = \mu(x_1) + \mu(z) + \gamma(y_2) = \mu(x_1) + \mu(z) + \gamma(y_2)$$

Since $z_1$ and $z_2$ are arbitrary, we get

$$\lambda(x + y) \leq \lambda(x) + \lambda(y)$$

Analogously, $\lambda'(x + y) \geq \lambda'(x) + \lambda'(y)$.

We show now that $\lambda \in C^\kappa$. One may easily see that $\lambda$ is additive on the whole $C$. Let $(x_n)$ be a sequence in $C$ such that $\sum x_n = x$. If $x \in C\mu \cap C\gamma$ we have $\lambda(x) = +\infty$ and
\[ \sum \lambda(x_n) \geq \sum \mu(x_n) = \mu(x), \quad \sum \lambda(x_n) \geq \sum \nu(x_n) = \nu(x) \]

hence \( \sum \lambda(x_n) = +\infty = \lambda(x) \). If \( x \in C_\mu \cap C_\nu \), we have

\[ \lambda(x) = \sum_{n=1}^{k} \lambda(x_n) + \lambda(\sum_{n=k+1}^{\infty} x_n) \]

and since

\[ \lim_{k \to \infty} \lambda(\sum_{n=k+1}^{\infty} x_n) = \lambda(x) \]

we have

\[ \lambda(x) = \sum_{n=1}^{\infty} \lambda(x_n) \]

Now, it results that \( \lambda \) is exactly \( \mu \lor \nu \) and the first part of the assertion c) is proved.

Using the above procedure of computing \( \lambda = \mu \lor \nu \), one may easily see that for any \( \mu' \in C_\mu \), we have

\[ \mu' \lor \lambda = (\mu' \lor \mu) \lor (\mu \lor \nu) \]

We prove now b) and e). Let \( (\mu_i)_{i \in J} \) be a family in \( C_\mu \). If \( J \) is finite, it is immediate by induction that \( \nu \lor \mu_i \) exists and

\[ \mu' \lor \lambda \lor \nu \lor \mu_i = \lor \nu (\mu' \lor \mu_i) \]

Now, if \( J \) is arbitrary, we may restrict ourselves to the case when the family \( (\mu_i)_{i \in J} \) is upper directed. Then the map \( x \mapsto \sup_{i} \mu_i(x) \) is exactly the \( \lor_{i} \mu_i \), hence \( C_\mu \) is a complete lattice. Also, the above distributivity formula, i.e. the first formula from e) holds.

We prove now the equality

\[ \mu' \lor \lambda \lor \nu \lor \mu_i = \lor \nu (\mu' \lor \mu_i) \]

Obviously, the left hand side is smaller, then the right. If we take \( \gamma \) equal to this one, we have \( \mu \leq \gamma \), hence there exists \( \gamma \in C_\mu \), such that \( \gamma = \gamma \lor \mu \). Let \( x \in C_{\mu} \). Then, from \( \mu' \lor \mu_i \geq \mu \lor \gamma \), one may easily show that we get \( \gamma \lor \mu_i x \leq \mu_i x \), hence

\[ \forall x \in C_{\mu} \leq \mu_i x \leq \mu_i x \]

which implies that the above equality holds on \( C_{\mu} \). On the complement